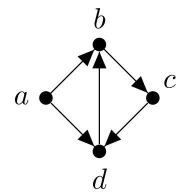
Counting Spanning Trees Using Determinants

Let $\vec{G}$ be an oriented graph. Let matrix D be its *incidence matrix* defined as

$$d_{ik} = \begin{cases} -1 & \text{if } i \text{ is the tail of } \vec{e}_k \\ 1 & \text{if } i \text{ is the head of } \vec{e}_k \\ 0 & \text{otherwise} \end{cases}$$

1: Find the incidence matrix D for the following oriented graph

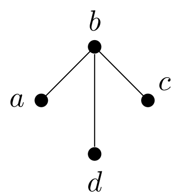


Solution:

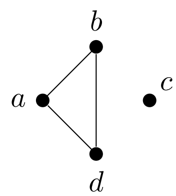
$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} D = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & -1 \end{pmatrix}$$

Let G be a graph on n vertices and $n-1$ edges. Let $\vec{G}$ be any orientation of G and let D be the incidence matrix of $\vec{G}$. Let $\overline{D}$ be obtained from D by deleting the first row corresponding to vertex x. Note that $\overline{D}$ is $(n-1)\times(n-1)$ matrix.

2: Find matrix $\overline{D}$ for

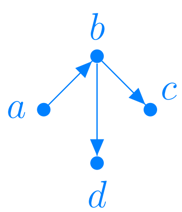


and

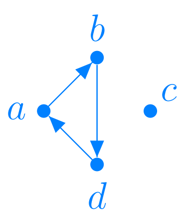


. Notice the solution is not unique.

Solution: In the solution, indexes are a, b, c, d and a is removed



gives $\overline{D} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$



gives $\overline{D} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$

Lemma 1 $\det \overline{D} \in \{-1, 0, 1\}$. Moreover $\det \overline{D} = 0$ iff G is not a tree.

3: Show that if G of order at least 3 has a leaf $v \neq x$ then $|\det \overline{D}| = |\det \overline{D'}|$, where D' corresponds to $\vec{G} - v$.

Solution: Row corresponding to v contains exactly one entry 1 or -1, which corresponds to the column of the edge incident to v. It is possible to expand the determinant using that row. This corresponds to deleting a leaf and its incident edge.

4: Show that if G is a tree then $\det \overline{D} \in \{-1, 1\}$.

Solution: Tree has at least two leaves. So the previous question applies and we can delete a leaf if G has at least 3 vertices. If G has just two vertices, it has just one edge and clearly $\det \overline{D} \in \{-1, 1\}$, since

$$D = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ or } D = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

5: Show that if x is an isolated vertex then $\det \overline{D} = 0$.

Solution: Sum of all the rows in $\overline{D}$ is 0, hence $\overline{D}$ does not have full rank hence $\det \overline{D} = 0$.

6: Show that if $v \neq x$ is an isolated vertex then $\det \overline{D} = 0$.

Solution: Then $\overline{D}$ contains a row of zeros so $\det \overline{D} = 0$.

7: Show that if $deg(v) \geq 2$ for all $v \neq x$ and $deg(x) \geq 1$ then G has too many edges.

Solution: The number of edges is $\frac{1}{2}\sum_v deg(v) \geq \frac{1}{2}(1 + 2(n-1)) = n - \frac{1}{2} > n - 1$.

8: Notice this finishes the proof of Lemma 1.

Solution: If a graph has a leaf other than x, we can remove it and use induction on the number of vertices. Then we know x is not isolated. If G is not a tree we end up with a graph that is not P_2 and either it has isolated vertex or all vertices have degree at least 2.

Crazy idea of computing the number of spanning trees of a general graph G. Take the incidence matrix of $\vec{G}$ and take all subsets of edges of size $n-1$ and compute absolute values of determinants of corresponding $\overline{D}$ and sum them up.

Theorem Binet-Cauchy Let A be an arbitrary matrix with $n-1$ rows and m columns. Then

$$det(AA^T) = \sum_I det(A_I)^2$$

where the sum is over all $(n-1)$-element subsets of $\{1, 2\ldots, m\}$; that is $I \in \binom{\{1,2\ldots,m\}}{n-1}$ or also $I \subseteq \{1, 2\ldots, m\}$ and $|I| = n-1$; and A_I is obtained from A by deleting all columns whose index is not in I.

9: Compute DD^T for D from question 1. Recall $D = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & -1 \end{pmatrix}$

Solution:

$$DD^T = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

Laplace matrix of a graph G of order n is $n \times n$ matrix Q where

$$q_{ij} = \begin{cases} -1 & \text{if } ij \in E(G) \\ deg(i) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

10: Show that for any orientation of G holds $Q = DD^T$.

Solution: Entry q_{ii} corresponds to the dot product of a row i of D with itself. Every entry is squared and they are summed, which gives the degree of vertex i. Dot product of two different rows is always -1 as it comes from the endpoints of one edge.

11: Let Q_{11} be obtained from Q by deleting the first row and the first column. Show that for any orientation of G holds $Q_{11} = \overline{D}\,\overline{D}^T$.

Solution: All the entries that are in Q_{11} are only coming from $\overline{D}\,\overline{D}^T$.

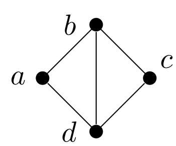
Matrix Tree Theorem Let G be a graph and Q its Laplace matrix. Then the number of spanning trees of G is equal to $\det(Q_{11})$ where Q_{11} is obtained from Q by deleting the first row and column.

12: Prove the Matrix Tree Theorem.

Solution: Notice it just follows from the previous claims.

$$\det(Q_{11}) = \det(\overline{D}\,\overline{D}^T) = \sum_I \det(\overline{D_I})^2$$

where I runs over all subsets of edges of G of size $n-1$. And $\det(\overline{D_I})^2 = 1$ if and only if I corresponds to a subset of edges that forms a spanning tree and otherwise $\det(\overline{D_I})^2 = 0$.

13: Count the number of spanning trees of 

using Matrix Tree Theorem.

Solution: Compute determinant of the matrix from question 8 with first row and column removed. Let Q be the Laplace matrix.

$$Q = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix} \text{ and } Q_{11} = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

Now we need to compute the determinant of Q_{11}:

$$\det(Q_{11}) = \begin{vmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{vmatrix} = \begin{vmatrix} 0 & -4 & 8 \\ 0 & 3 & -4 \\ -1 & -1 & 3 \end{vmatrix} = -\begin{vmatrix} -4 & 8 \\ 3 & -4 \end{vmatrix} = -(16-24) = 8$$

Hence there are 8 different spanning trees.

14: Count the number of spanning trees of K_n using Matrix Tree Theorem.

Solution: Sum rows and columns to obtain a nice way to compute the determinant. n^{n-2}.

15: Describe graphs that have exactly 3 spanning trees.

Solution: Graph that has exactly one cycle and the cycle is a triangle. It can be obtained from a tree by adding one edge between vertices of distance 2.

16: Show that if a graph G of order n is connected then it's Laplace matrix Q has rank $n-1$.

Solution: If the graph is not connected, it has no spanning trees and hence the determinant of Q is zero.

17: Use Cayley's formula to prove that the graph obtained from K_n by deleting an edge has $(n-2)n^{n-3}$ spanning trees.

Solution: Count how many times is each edge used in a tree. Since K_n is so symmetric, every edge is in the same number of trees. Conclusion is that one edge is in $2n^{n-3}$ trees.

18: Count the number of spanning trees of $K_{m,n}$.

Solution:

19: *Open Problem* Prove or disprove: Let G be a graph with the minimum vertex degree at least 2; that is, $\delta(G) \geq 2$. Then there exists a spanning tree T of G such that for every vertex v in T that is adjacent to a leaf in T the following holds if $\deg_G(v) \geq 3$, then $\deg_T(v) \geq 3$.